Stirling's Approx:

Desired result $\ln(N!) \approx N\ln N - N$ for large N

Eg: $\ln(50!) = 148.4$

$50\ln(50) - 50 = 145.6$

Simple Derivation:

$$\ln(N!) = \ln\left[N(N-1)(N-2)\ldots(1)\right]$$

$$= \ln(N) + \ln(N-1) + \ln(N-2) + \ldots + \ln(1)$$

$$= \sum_{x=1}^{N} \ln(x)$$

Approx. sum as an integral

$$\ln(N!) = \sum_{x=1}^{N} \ln(x) \approx \int_1^N \ln(x)\,dx$$

Integrate $\ln(x)$ by parts. Recall $\int u\,dv = uv - \int v\,du$

take $u = \ln x$ $\quad dv = dx$

$\therefore du = \frac{dx}{x}$ $\quad \therefore v = x$

$uv = x\ln x$

$v\,du = x\frac{dx}{x} = dx$

$$\int_1^N \ln(x)\,dx = x\ln x\Big|_1^N - \int_1^N dx$$

$$\therefore \ln(N!) \approx N\ln N - \underbrace{\ln(1)}_{0} - \underbrace{(N-1)}_{\approx N}$$

$$\boxed{\therefore \ln(N!) \approx N\ln N - N}$$

Can do better with a little more work.

Can write factorials in terms of the Γ fcn.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = (z-1)!$$

$$\therefore \Gamma(z+1) = \boxed{\int_0^\infty e^{-t} t^z dt = z!}$$

we first verify the expression in the box.

Evaluate $\int_0^\infty e^{-t} t^z dt$ using integration by parts.

take $u = t^z$ $\quad dV = e^{-t} dt$

$\therefore du = z t^{z-1} dt$ $\quad \therefore V = -e^{-t}$

$$\int_0^\infty e^{-t} t^z dt = \underbrace{-e^{-t} t^z \Big|_0^\infty}_{} - \int_0^\infty (-e^{-t}) z t^{z-1} dt$$

e^{-t} goes to zero faster than $t^z \to \infty$ for large t (use l'Hopital's rule)

This term goes to zero

$$= z \underbrace{\int_0^\infty e^{-t} t^{z-1} dt}_{\Gamma(z)}$$

$$\therefore \Gamma(z+1) = z\,\Gamma(z)$$

Integrate by parts again:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$$

$u = t^{z-1}$ $\quad dV = e^{-t} dt$

$\therefore du = (z-1) t^{z-2} dt \quad \therefore V = -e^{-t}$

$$\Gamma(z) = \underbrace{-e^{-t} t^{z-1}\Big|_0^\infty}_{0} - \int_0^\infty (-e^{-t})(z-1) t^{z-2}\, dt$$

$$= (z-1)\int_0^\infty e^{-t} t^{z-2}\, dt$$

$$= (z-1)\,\Gamma(z-1)$$

$\therefore$ Have shown so far that

$$\Gamma(z+1) = z(z-1)\,\Gamma(z-1)$$

can continue in this fashion. If take z to be integer, eventually get to:

$$\Gamma(z+1) = z(z-1)(z-2)\cdots(2)(1)\,\Gamma(1)$$

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t}\Big|_0^\infty = \underbrace{-e^{-\infty}}_{0} - (\underbrace{-e^{0}}_{-1}) = 1$$

So we have managed to show that:

$$\Gamma(z+1) \equiv \boxed{\int_0^\infty e^{-t}\, t^z\, dt = z!}$$

Now back to Stirling's Approx.

$$N! = \int_0^\infty e^{-t}\, t^N dt$$

note that $t^N = e^{N\ln t}$

$$\therefore N! = \int_0^\infty e^{N\ln t - t}\, dt$$

make a substitution $t = N + N^{1/2}x$

$$\therefore dt = N^{1/2}dx$$

limits:

when $t = 0$, $x = -N^{1/2}$

$t = \infty$, $x = \infty$

$$N! = \int_{x=-\sqrt{N}}^{\infty} \text{Exp}\left\{ N\ln(N+\sqrt{N}\,x) - N - \sqrt{N}\,x \right\} \sqrt{N}\,dx$$

$$= \int_{-\sqrt{N}}^{\infty} \text{Exp}\left\{ N\ln\left[N\left(1+\frac{x}{\sqrt{N}}\right)\right] - N\left(1+\frac{x}{\sqrt{N}}\right)\right\} \sqrt{N}\,dx$$

$$N! = \sqrt{N}\int_{-\sqrt{N}}^{\infty} \text{Exp}\left\{N\ln N + N\ln\left(1+\frac{x}{\sqrt{N}}\right) - N\left(1+\frac{x}{\sqrt{N}}\right)\right\}dx$$

$$= \sqrt{N}\,N^N e^{-N}\int_{-\sqrt{N}}^{\infty} \text{Exp}\left\{N\ln\left(1+\frac{x}{\sqrt{N}}\right) - \sqrt{N}\,x\right\}dx$$

Now we make our first approx.

Recall that $\ln(1+\delta) \approx \delta - \frac{\delta^2}{2} + \dots$

since N is large (or we take N to be large) we can approx

$$\ln\left(1+\frac{x}{\sqrt{N}}\right) \approx \frac{x}{\sqrt{N}} - \frac{x^2}{2N}$$

s.t. $N\ln\left(1+\frac{x}{\sqrt{N}}\right) - \sqrt{N}\,x$

$$\approx \sqrt{N}\,x - \frac{x^2}{2} - \sqrt{N}\,x = -\frac{x^2}{2}$$

$$\therefore N! \approx \sqrt{N}\,N^N e^{-N}\int_{-\sqrt{N}}^{\infty} e^{-x^2/2}\,dx$$

Our next approx is to take $-\sqrt{N}$ as $-\infty$ in integration limit. Not too bad of an approx since $e^{-x^2/2}$ very quickly goes to zero away from $x=0$.

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx \quad \text{is a Gaussian integral}$$
$$= \sqrt{2\pi}$$

$$\boxed{\therefore\ N! \approx \sqrt{2\pi N}\, N^N e^{-N} \quad \text{Stirling's Approx}}$$

$$\therefore\ \ln(N!) \approx \ln\left(\sqrt{2\pi N}\, N^N e^{-N}\right)$$

$$\boxed{\ln(N!) = N\ln N - N + \tfrac{1}{2}\ln(2\pi N)}$$

check $N = 50$

$$50\ln 50 - 50 + \tfrac{1}{2}\ln(2\pi\cdot 50) = 148.5$$ very close to actual answer!

Actually, this form of Stirling's approx. is pretty good even for small nos.

$$\ln(3!) = \ln(6) = 1.79$$
$$3\ln 3 - 3 + \tfrac{1}{2}\ln(2\pi\cdot 3) = 1.76$$

For very large nos., as are common in Stat. Mech, the $\ln(2\pi N)$ term is small c.t. the other two terms s.t.

$$\boxed{\ln(N!) \approx N\ln N - N}$$